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PROPAGATION OF A BRITTLE CRACK AT CONSTANT AND ACCELERATING SPEEDS

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Abstract—The integral solutions for a symmetrical crack propagating at a varying speed in an elastic solid under the action of an arbitrary symmetrical crack pressure are obtained as sums of associated static solutions and stress-wave integrals. Two special cases are studied in detail. For the case of a symmetrical crack running at a constant speed under a uniform pressure, exact dynamic solutions for crack shape and stress distribution with singularities in the crack plane are obtained in closed forms that are easily comparable to associated static solutions. The difference between dynamic and static solutions for quantities such as crack shape and stress intensity factors is governed by dynamic correction factors which are nondimensional functions of Poisson's ratio and the ratio between crack speed and shear-wave speed. The values of these dynamic factors are obtained for large range of crack speed and the difference can clearly be determined from the results obtained.

A study is made for the propagation of a crack at a constant acceleration. The quantities similar to those obtained for the above constant-speed crack are also obtained. The deviations of crack-shape and stress-intensity factors from the associated static state are relatively smaller for an accelerating crack than for a constant-speed crack propagating at the same speed.

1. INTRODUCTION

A CRACK was observed to be growing from rest at a varying velocity [1, 2]. Theoretical investigations on crack dynamics were recently carried out by a number of workers [3–5, 13–19]. An exact solution was obtained by the author for a penny-shaped crack propagating at a constant speed in an infinite elastic solid [4]. The solution obtained by Broberg [5] for a constant-speed plane crack met the condition of vanishing normal acceleration instead of directly satisfying the usual condition of vanishing normal displacement in the crack plane.

Using the techniques developed in an earlier paper [4], the problem for a brittle plane crack propagating at a varying speed is considered in the present work. The crack is assumed to be propagating along its own plane. Laplace and Fourier transforms are used to solve the equations of motion and satisfy the dynamic boundary conditions. For a constant-speed crack, the crack shape and the stress distribution in the crack plane are explicitly obtained in exact expressions easily comparable to the associated static solutions [6]. The dynamic "stress intensity" functions are obtained in terms of crack speed, wave speeds and Poisson's ratio. Solutions for quantities mentioned above are also obtained for an accelerating crack. Comparisons are made between the results for constant-speed and accelerating cracks.

2. CRACK OF ARBITRARY SPEED

Consider a two-dimensional elastic solid subjected to equilibrate, symmetrical tensions $\sigma(x)$ at infinity. A crack starts to propagate at $t = 0$ with a varying speed and has length

$2a(t)$ in a plane, $y = 0$ perpendicular to the direction of tension. Solution of the problem can be obtained by superposition of the tension field $\sigma(x)$ and the stress fields set up by a pressure $-\sigma(x)$ acting on the crack surfaces. The latter problem is the main subject of interest in the present work. The dynamic boundary conditions concerned can be prescribed as on the crack plane $y = 0$ for $t > 0$

$$\sigma_{xy} = 0 \tag{1}$$

and

$$v = \begin{cases} w(x, t), x \leq a(t) \\ 0, x > a(t) \end{cases} \tag{2}$$

where v is the vertical displacement normal to the crack plane and $w(x, t)$ is an unknown crack-shape function to be determined later. The equations of motion to be satisfied for a homogeneous, isotropic solid are

$$(\lambda+2\mu)\nabla(\nabla\,.\,\mathbf{u})-\mu\nabla\times(\nabla\times\mathbf{u}) = \rho\frac{\partial^2 u}{\partial t^2} \tag{3}$$

where the displacement vector $\mathbf{u}$ has the horizontal component u and the vertical component v; λ and μ are Lame's constants and ρ the density of the medium. The dilatation is

$$\Delta = \nabla\,.\,\mathbf{u} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \tag{4}$$

and the only nonvanishing component of the rotation is

$$\Omega = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{5}$$

Equations of motion (3) are satisfied if the following two wave equations are satisfied:

$$\nabla^2\Delta = \frac{1}{c_1^2}\frac{\partial^2\Delta}{\partial t^2}, \qquad c_1^2 = \frac{\lambda+2\mu}{\rho} \tag{6}$$

and

$$\nabla^2\Omega = \frac{1}{c_2^2}\frac{\partial^2\Omega}{\partial t^2}, \qquad c_2^2 = \frac{\mu}{\rho}. \tag{7}$$

To solve the equations, Laplace transforms $f^*(p)$ are operated over the time t. Furthermore, Fourier cosine or sine transforms are applied over x and defined as

$$\begin{matrix}\bar{f}_c \\ \bar{f}_s\end{matrix} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\begin{matrix}\cos(sx) \\ \sin(sx)\end{matrix}\,\mathrm{d}x. \tag{8}$$

Proper transformations reduce equation (6) and equation (7) to simple ordinary equations of y. Their solutions, with vanishing values at infinity, are found as

$$\bar{\Delta}_c^* = A\,\mathrm{e}^{-\alpha y} \quad \text{and} \quad \bar{\Omega}_s^* = B\,\mathrm{e}^{-\beta y} \tag{9}$$

where

$$\begin{aligned} &\alpha^2 = s^2+k_1^2, \qquad \beta^2 = s^2+k_2^2, \\ &k_1 = p/c_1 \quad \text{and} \quad k_2 = p/c_2. \end{aligned} \tag{10}$$

The transformed equations for displacements and stresses are found as

$$\frac{\rho p^2}{\mu}\bar{u}_s^* = -k^2 s\bar{\Delta}_c^* - \frac{\partial \bar{\Omega}_s^*}{\partial y} \tag{11}$$

$$\frac{\rho p^2}{\mu}\bar{v}_c^* = k^2\frac{\partial \bar{\Delta}_c^*}{\partial y} + s\bar{\Omega}_s^*. \tag{12}$$

$$_c\bar{\sigma}_{xx}^* = \lambda\bar{\Delta}_c^* + 2\mu s\bar{u}_s^* \tag{13}$$

$$_s\bar{\sigma}_{xy}^* = \mu\left(\frac{\partial \bar{u}_s^*}{\partial y} - s\bar{v}_c^*\right) \tag{14}$$

and

$$_c\bar{\sigma}_{yy}^* = \lambda\bar{\Delta}_c^* + 2\mu\frac{\partial \bar{v}_c^*}{\partial y}, \tag{15}$$

where $k^2 = k_2^2/k_1^2 = 2(1-\nu)/(1-2\nu)$ and ν is Poisson's ratio. Satisfying boundary conditions (1) and (2) yields

$$-\alpha k^2 A + sB = \frac{\rho p^2}{\mu}\bar{w}_c^* \tag{16}$$

and

$$2\alpha k^2 sA - (\beta^2 + s^2)B = 0. \tag{17}$$

Solutions of (16) and (17) are

$$A = -\frac{2s^2 + k_2^2}{\alpha k^2}\bar{w}_c^* \tag{18}$$

and

$$B = -2s\bar{w}_c^*. \tag{19}$$

The transformed stress $_c\bar{\sigma}_{yy}^*$ on the crack plane can now be calculated to be

$$\begin{aligned} _c\bar{\sigma}_{yy}^* &= -\frac{\mu}{k_2^2}\frac{(2s^2+k_2^2)^2 - 4s^2\alpha\beta}{\alpha}\bar{w}_c^* \\ &= -\mu k_2[k_2/\alpha + F^*(s,p)]\bar{w}_c^*, \end{aligned} \tag{20}$$

where

$$F^*(s,p) = \frac{4s^2(s^2 + k_2^2 - \alpha\beta)}{ak_2^3}. \tag{21}$$

The expression for $_c\bar{\sigma}_{yy}^*$ in equation (20) is the same as that for the similar normal stress in the earlier paper [4]. After a proper contour integration, the Laplace inversion of equation (20) is the same as that for the similar normal stress [4]. The inversion is then operated upon by Fourier inversions to obtain on $y = 0$

$$\sigma_{yy}(x,t) = \sigma_{yy}^0 - \rho c_1 Q_1 - \mu L_1 Q_2, \tag{22}$$

where

$$\sigma_{yy}^{0} = -K\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\cos(sx)s\bar{w}_c\,\mathrm{d}s \tag{23}$$

$$Q_1 = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\cos(sx)\frac{\partial}{\partial t}\int_{0}^{t}J_0(sc_1(t-\tau))\frac{\partial}{\partial \tau}\bar{w}_c\,\mathrm{d}\tau\,\mathrm{d}s \tag{24}$$

and

$$Q_2 = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\cos(sx)s\int_{0}^{t}\cos(sc_2\eta(t-\tau))\frac{\partial}{\partial \tau}\bar{w}_c\,\mathrm{d}\tau\,\mathrm{d}s. \tag{25}$$

The constant K is defined as $K = \mu/(1-\nu) = E/2(1-\nu^2)$. The operator L_1 resulting from contour integration is defined in Appendix A, and operates over η in equation (25) [4]. The first term σ_{yy}^{0} in equation (22) will soon be shown to be in the associated normal stress for a static crack. The second and third terms apparently account for the effect of stress waves. Equation (22) will result in an integral equation for the unknown $w(x, t)$. The value of σ_{yy} is known as $\sigma(x, t)$ for $x \leq a(t)$, but is unknown for $x > a(t)$. The value $\sigma(x, t)$ will be used to determine the unknown function $w(x, t)$. To solve for the dynamic crack-shape function w, the associated static crack-shape function w^0 will first be determined from equation (22) by dropping the last two wave integrals in equation (22). The reduced equation becomes

$$\begin{aligned}\sigma_{yy} &= -\frac{2K}{\pi}\int_{0}^{\infty}\cos(sx)s\int_{0}^{a}w^0\cos(sm)\,\mathrm{d}m\,\mathrm{d}s \\ &= \frac{2K}{\pi}\int_{0}^{\infty}\cos(sx)\int_{0}^{a}\frac{\partial w^0}{\partial m}\sin(sm)\,\mathrm{d}m\,\mathrm{d}s.\end{aligned} \tag{26}$$

The condition that the crack tip has vanishing normal displacement is used in equation (26), i.e. $w^0(a, t) = 0$. To solve for $w^0(x, t)$, the variable x is first changed to λ in equation (26), and the equation is then multiplied by the function $(x^2-\lambda^2)^{-\frac{1}{2}}$ with square-root singularity and finally integrated over λ [4]. Since $\sigma_{yy} = \sigma$ for $x \leq a$, the following integrals are obtained from equation (26):

$$\int_{0}^{x}\frac{\sigma\,\mathrm{d}\lambda}{\sqrt{(x^2-\lambda^2)}} = K\int_{0}^{a}\frac{\partial w^0}{\partial m}\int_{0}^{\infty}J_0(sx)\sin(sm)\,\mathrm{d}s\,\mathrm{d}m = K\int_{x}^{a}\frac{\partial w^0}{\partial m}\frac{1}{\sqrt{(m^2-x^2)}}\mathrm{d}m. \tag{27}$$

A further integration over equation (27) as indicated yields

$$w^0(x, t) = -\frac{2}{\pi K}\int_{x}^{a}\frac{n\,\mathrm{d}n}{\sqrt{(n^2-x^2)}}\int_{0}^{n}\frac{\sigma\,\mathrm{d}\lambda}{\sqrt{(n^2-\lambda^2)}}. \tag{28}$$

This is a general expression for the associated static crack-shape function produced by a prescribed crack pressure σ. For the special case of a uniform crack pressure, i.e., $\sigma_{yy} = -p_0$ for $x < a$, the crack function will be integrated out from equation (28) to be the same as that for a static crack [6]. If the operations on equation (28) are applied to equation (22), the dynamic equation for $w(x, t)$ is obtained from (22) as

$$w(x, t) = w^0(x, t) - \frac{2}{\pi K}\int_{x}^{a}\frac{n\,\mathrm{d}n}{\sqrt{(n^2-x^2)}}\int_{0}^{n}\frac{\rho c_1 Q_1 + \mu L_1 Q_2}{\sqrt{(n^2-\lambda^2)}}\mathrm{d}\lambda\,\mathrm{d}n. \tag{29}$$

This is a general equation to determine $w(x, t)$ for a prescribed crack pressure σ. If w is determined from equation (29), the results may be substituted into equation (22) to obtain the normal stress $\sigma_{yy}(x, t)$. The other nonvanishing stress σ_{xx} on the crack plane can be obtained by using the same procedures as that for σ_{yy}. The transformed solution is found as

$$ {}_c\bar{\sigma}^*_{xx} = \frac{\mu}{k_2^2}\left[\frac{4s^2(s^2+k_1^2-\alpha\beta)}{\alpha} - \frac{v}{1-v}\frac{k_2^4}{\alpha}\right]\bar{w}^*_c. \tag{30} $$

The procedures used for inverse transforms of equation (30) are similar to those for σ_{yy}. The results obtained are

$$ \sigma_{xx} = \sigma^0_{xx} - \frac{v}{1-v}\rho c_1 Q_1 - \mu \bar{L}_1 Q_2 \tag{31} $$

where the associated static stress is identical with σ^0_{yy} in equation (23), i.e.,

$$ \sigma^0_{xx} = \sigma^0_{yy}. \tag{32} $$

The new operator $\bar{L}_1$ that resulted from inversions is defined in Appendix A. The expressions and intervals of integration in the operators L_1 and $\bar{L}_1$ are similar to those obtained for vertical displacements [7–10]. All the equations (22), (29) and (31) are written as sums of associated static expressions and wave-effect integrals for an arbitrary symmetrical pressure $\sigma(x, t)$. If the integrations involved can be carried out, a clear comparison can be made between dynamic and static quantities. In all the above calculations, the constancy of crack speed is not required; therefore, the results are valid for both constant and varying crack speeds. In the following sections, results for a constant-speed crack are obtained in closed forms and solutions for an accelerating crack are also obtained.

3. CONSTANT-SPEED CRACK

Closed-form solutions are obtained if the crack is assumed to propagate at a constant speed V with uniform pressure $-p_0$ on the crack surfaces. To obtain the solution, a method of successive approximation is used [4, 11, 12]. The first term on the right-hand side of equation (29) gives the first approximation of $w(x, t)$. Thus, for the uniform pressure prescribed, we have

$$ w^1(x, t) = w^0(x, t) = \frac{p_0}{K}\sqrt{(a^2-x^2)}. \tag{33} $$

This is precisely the shape of an associated static crack [6]. To obtain the next approximation of equation (29), integrals (24) and (25) have to be carried out. From equation (33), the time derivative of the cosine transform w^0 is found as

$$ \frac{\partial}{\partial t}\bar{w}^0_c = \frac{p_0}{K}\sqrt{\frac{\pi}{2}}J_0(sa)a\dot{a} \tag{34} $$

where the dot means differentiation with respect to t. If the following identity is used:

$$ J_0[sc_1(t-\tau)] = \frac{2}{\pi}\int_0^\infty \sin[sc_1 \cosh \xi(t-\tau)]\,\mathrm{d}\xi \tag{35} $$

the first approximation of Q_1 in equation (24) for $x \leq a$ in terms of w^1 is found as

$$
\begin{aligned}
Q_1^1 &= \frac{2p_0}{\pi K}\frac{\partial}{\partial t}\int_0^\infty \int_0^t \int_0^\infty \cos(sx)\sin(sc_1\cosh\xi(t-\tau))J_0(sa)a\dot{a}\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&= \frac{p_0}{\pi K}\frac{\partial}{\partial t}\int_0^\infty \int_0^t \int_0^\infty \{\sin s[c_1\cosh\xi(t-\tau)+x] \\
&\quad +\sin s[c_1\cosh\xi(t-\tau)-x]\}J_0(sa)a\dot{a}\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&= \frac{p_0}{\pi K}\frac{\partial}{\partial t}\int_0^\infty \int_0^t \{G_3H[c_1\cosh\xi(t-\tau)+x-a] \\
&\quad +G_2H[c_1\cosh\xi(t-\tau)-x-a]\}a\dot{a}\,\mathrm{d}\tau\,\mathrm{d}\xi
\end{aligned}
\tag{36}
$$

where H is the Heaviside function and the wave functions are

$$G_3(\cosh\xi, x, t) = \{[c_1\cosh\xi(t-\tau)+x]^2-a^2\}^{-\frac{1}{2}} \tag{37}$$

and

$$G_2(\cosh\xi, x, t) = \{[c_1\cosh\xi(t-\tau)-x]^2-a^2\}^{-\frac{1}{2}}. \tag{38}$$

For a crack of constant speed V as prescribed, $\dot{a} = V$ and $a = V\tau$. The integration of the first term over τ in equation (36) may be written as

$$
\begin{aligned}
\int_0^{\tau_4} G_3\tau\,\mathrm{d}\tau &= \int_0^{\tau_4} [(c_1t\cosh\xi+x)-(c_1\cosh\xi-V)\tau]^{-\frac{1}{2}} \\
&\quad .\,[(c_1t\cosh\xi+x)-(c_1\cosh\xi+V)\tau]^{-\frac{1}{2}}\tau\,\mathrm{d}\tau
\end{aligned}
\tag{39}
$$

where the upper limit of integration due to H is

$$\tau_4 = \frac{c_1t\cosh\xi+x}{c_1\cosh\xi+V}. \tag{40}$$

Using formula (a) in Appendix B, equation (39) can be integrated explicitly. The second term in equation (36) can similarly be integrated out with the upper integration limit as

$$\tau_2 = \frac{c_1t\cosh\xi-x}{c_1\cosh\xi+V}. \tag{41}$$

The above integrations followed by differentiation with respect to t of equation (36) give

$$Q_1^1 = \frac{p_0}{K}c_1D_1 \tag{42}$$

where the constant D_1 is defined in Appendix C as a function of $v_1 = V/c_1$.

Similarly, the first approximation of Q_2 in equation (25) for $x \leq a$ is found as

$$
\begin{aligned}
Q_2^1 &= \frac{p_0}{2K}\frac{\partial}{\partial x}\int_0^t \int_0^\infty \{\sin s[c_2\eta(t-\tau)+x]-\sin s[c_2\eta(t-\tau)-x]\}J_0(sa)a\dot{a}\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \frac{p_0}{2K}\frac{\partial}{\partial x}\left\{\int_0^{\tau_4} \mathbf{G}_3(\eta, x, t)a\dot{a}\,\mathrm{d}\tau - \int_0^{\tau_2} \mathbf{G}_2(\eta, x, t)a\dot{a}\,\mathrm{d}\tau\right\}.
\end{aligned}
\tag{43}
$$

All the functions such as $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_4$, $\mathbf{G}_3$ and $\mathbf{G}_2$ are respectively equal to the corresponding functions τ_2, τ_4, G_3 and G_2 if $\cosh\xi$ is replaced by η. This convention will be applied hereafter. The above equation can further be integrated as before to be

$$L_1 Q_2^1 = \frac{p_0}{K} D_2 \tag{44}$$

where the constant D_2 is defined in Appendix C in terms of the velocities ratio $v_2 = V/c_2$. If equations (42) and (44) are substituted in equation (29), the second approximation of w is found as

$$w^2(x, t) = w^0[1-\varepsilon] \tag{45}$$

where the dynamic correction constant is

$$\varepsilon = (1-\nu)(k^2 D_1 + D_2). \tag{46}$$

If the successive approximations continue, the solution for w from equation (29) results in an infinite series and can be written in a closed form as follows:

$$\begin{aligned} w(x, t) &= w^0(x, t)(1-\varepsilon+\varepsilon^2-\varepsilon^3+\ldots) \\ &= \frac{w^0(x, t)}{1+\varepsilon} = \frac{p_0}{K_D}\sqrt{(a^2-x^2)} \end{aligned} \tag{47}$$

where the dynamic constant is

$$K_D = K(1+\varepsilon). \tag{48}$$

If crack speed V tends toward zero, the dynamic correction term ε vanishes. Therefore, the crack-shape function (47) reduces to that for a static crack.

The exact expression for w in equation (47) may be used to determine the stress distribution in the crack plane. To find the normal stress in equation (22), the cosine transform of w is found as

$$\bar{w}_c = \frac{p_0}{K_D}\sqrt{\frac{\pi}{2}}\frac{a}{s}J_1(sa). \tag{49}$$

The associated static normal stress in equation (23) is found in terms of $\bar{w}_c$ as

$$\sigma_{yy}^0 = -\frac{p_0}{1+\varepsilon} a \int_0^\infty \cos(sx) J_1(sa)\,\mathrm{d}s. \tag{50}$$

This is integrated out as

$$\sigma_{yy}^0 = \begin{cases} -\dfrac{p_0}{1+\varepsilon} & \text{for } x \le a \\[2ex] \dfrac{p_0}{1+\varepsilon}\left[\dfrac{x}{\sqrt{(x^2-a^2)}} - 1\right] & \text{for } x > a. \end{cases} \tag{51}$$

When the value of ε tends to zero for vanishing V, equation (51) reduces to the same static normal stress as that found through a different process [6]. The values for the wave-effect integrals Q_1 and Q_2 in equation (22) for $x \le a$ are respectively equal to Q_1^1 in equation (42) and Q_2^1 in equation (44) if K is replaced by K_D. In terms of these values, the normal stress σ_{yy} completely recovers its prescribed value $-p_0$ for $x \le a$. If the procedures similar to those in equation (36) are used, the integral Q_1 for $a < x < c_1 t$ is found as

$$Q_1 = \frac{p_0}{\pi K_D}\frac{\partial}{\partial t}\int_0^\infty \left\{\int_0^t G_3(\cosh\xi, x, t)a\dot{a}\,\mathrm{d}\tau + \int_0^{\tau_2} G_2(\cosh\xi, x, t)a\dot{a}\,\mathrm{d}\tau - \int_{\tau_1^1}^t G_1(\cosh\xi, x, t)a\dot{a}\,\mathrm{d}\tau\right\}\mathrm{d}\xi \tag{52}$$

where

$$G_1(\cosh\xi, x, t) = \{[x - c_1\cosh\xi(t-\tau)]^2 - a^2\}^{-\frac{1}{2}} \tag{53}$$

and

$$\tau_1^1 = \frac{c_1 t\cosh\xi - x}{c_1\cosh\xi - V}. \tag{54}$$

All the integrals in equation (52) can be reduced to the forms in Appendix B and integrated out explicitly. After a lengthy calculation, the results can be written as

$$Q_1 = \frac{p_0 c_1}{K_D k^2}\left\{D_3\left(\frac{x}{\sqrt{(x^2-a^2)}} - 1\right) + Q_{12}(x, t)\right\} \tag{55}$$

where

$$Q_{12}(x, t) = L_2 \ln\left\{\frac{\sqrt{(c_1^2 t^2\cosh^2\xi - x^2)}[1 + F(\cosh\xi, v_1)]}{x + F(\cosh\xi, v_1)\sqrt{(x^2-a^2)}}\right\}. \tag{56}$$

The function used is

$$F(\cosh\xi, v_1) = \cosh\xi(\cosh^2\xi - v_1^2)^{-\frac{1}{2}}. \tag{57}$$

The dynamic constant D_3 is defined in Appendix C whereas the operator L_2 over ξ is defined in Appendix A. The integral Q_2 in equation (25) for $a < x < c_2 t$ may be written by the similar procedures used in equations (36) and (43) to be

$$Q_2 = \frac{p_0}{2K_D}\frac{\partial}{\partial x}\left\{\int_0^t G_3(\eta, x, t)a\dot{a}\,\mathrm{d}\tau - \int_0^{\tau_2} G_2(\eta, x, t)a\dot{a}\,\mathrm{d}\tau + \int_{\boldsymbol{\tau}_1^1}^t G_1(\eta, x, t)a\dot{a}\,\mathrm{d}\tau\right\} \tag{58}$$

where $\boldsymbol{\tau}_1^1$ is equal to τ_1^1 in equation (54) if $\cosh\xi$ is replaced by η. Integral (58) has situations similar to those for equation (52) and is integrated out explicitly to be

$$L_1 Q_2 = \frac{p_0}{K_D}\left\{D_4\left(\frac{x}{\sqrt{(x^2-a^2)}} - 1\right) + Q_{22}(x, t)\right\} \tag{59}$$

where

$$Q_{22} = L_3 \ln\frac{\sqrt{(c_2^2 t^2\eta^2 - x^2)}[1 + F(\eta, v_2)]}{x + F(\eta, v_2)\sqrt{(x^2-a^2)}} \tag{60}$$

The dynamic constant D_4 is defined in Appendix C whereas the operator L_3 over η is defined in Appendix A. If equations (51), (55) and (59) are substituted in equation (22), the normal stress on the crack plane for $a < x < c_2 t$ is obtained as

$$\sigma_{yy}(x, t) = p_0\left\{K_{Iv}\left(\frac{x}{\sqrt{(x^2-a^2)}} - 1\right) - \frac{(1-v)}{1+\varepsilon}(Q_{12} + Q_{22})\right\} \tag{61}$$

where the dynamic "stress intensity" correction factor in the cleavage plane is

$$K_{Iv} = [1-(1-\nu)(D_3+D_4)]/(1+\varepsilon). \tag{62}$$

The dynamic stress intensity factor is equal to the above constant multiplied by the associated static stress intensity factor, e.g., $K_{Iv}p_0\sqrt{(\pi a)}$. The stress σ_{xx} in equation (31) can also be obtained if the procedures for determining σ_{yy} are followed.

$$\sigma_{xx} = \begin{cases} -p_0\dfrac{1+\nu k^2 D_1+(1-\nu)\bar{D}_2}{1+\varepsilon} & \text{for } |x| \le a \\ p_0\left\{K_{Iv}^1\left[\dfrac{x}{\sqrt{(x^2-a^2)}}-1\right]-[\nu Q_{12}+(1-\nu)\bar{Q}_{22}]/(1+\varepsilon)\right\} & \text{for } a < |x| < c_2 t \end{cases} \tag{63}$$

where the dynamic "stress intensity" correction factor in the noncleavage plane is

$$K_{Iv}^1 = [1-(\nu D_3+(1-\nu)\bar{D}_4)]/(1+\varepsilon). \tag{64}$$

The above functions $\bar{D}_2$, $\bar{D}_4$ and $\bar{Q}_{22}$ are respectively equal to D_2, D_4 and Q_{22} if L_1 in the operators involved is replaced by $\bar{L}_1$. If the crack speed V tends to zero giving $v_1 = v_2 = 0$, the stresses σ_{yy} and σ_{xx} simply reduce to their corresponding static stresses [6].

4. ACCELERATING CRACK

The integral equation (29) for determining $w(x, t)$ holds for a crack propagating at a varying speed. Therefore, it is good for a crack which starts to propagate at $t = 0$ with a constant acceleration γ. This condition gives a crack speed $V = \gamma t$ and a half crack length $a = \frac{1}{2}\gamma t^2$. To solve for w in equation (29), the previous method of successive approximations will be used. The first approximation of w is exactly the same as that in equation (33). Consequently, equation (36) also holds. However, the wave functions G_3 and G_2 are here different from the previous functions of τ. Under the present condition, equation (38) leads to

$$G_2^{-2}(\cosh\xi, x, t, \tau) = z_1(\cosh\xi, x, t, \tau)z_2(\cosh\xi, x, t, \tau) \tag{65}$$

where

$$\begin{aligned} z_1 &= c_1\cosh\xi(t-\tau)-x+a(\tau) \\ &= (\tau-t_1)(\tau-t_1^1)\gamma/2 \end{aligned} \tag{66}$$

and

$$\begin{aligned} z_2 &= c_1\cosh\xi(t-\tau)-x-a(\tau) \\ &= (t_2-\tau)(\tau-t_1^1)\gamma/2. \end{aligned} \tag{67}$$

In the above equations, the roots of z_1 and z_2 are

$$\left.\begin{matrix} t_1 \\ t_1^1 \end{matrix}\right\} = \{\pm\sqrt{[c_1^2\cosh^2\xi-2\gamma(c_1\cosh\xi t-x)]}+c_1\cosh\xi\}/\gamma \tag{68}$$

and

$$\left.\begin{matrix} t_2 \\ t_2^1 \end{matrix}\right\} = \{\pm\sqrt{[c_1^2\cosh^2\xi+2\gamma(c_1\cosh\xi t-x)]}-c_1\cosh\xi\}/\gamma. \tag{69}$$

field, ψ, which is a function of the state variables defining the system and their time derivatives. With the aid of the governing differential equations of motion, the integral of ψ over the domain is then evaluated to first order in the functional neighborhood of the steady state. It is then possible to separate the time differentiation from the integration, obtaining

$$\int \psi \, \mathrm{d}v = \frac{\partial \Phi}{\partial t} \leq 0$$

where Φ is termed a local potential and is in the nature of a generalized rate of entropy production. For Beck's problem, Φ may be defined as

$$\Phi(u, u^0) = \int_0^1 \left\{ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u^0}{\partial x^2} + Fu\frac{\partial^2 u^0}{\partial x^2} + \gamma \frac{\partial u^0}{\partial t} u + m \frac{\partial^2 u^0}{\partial t^2} u \right\} \mathrm{d}x$$

so that

$$\delta\Phi = \int_0^1 \left(\frac{\partial^4 u^0}{\partial x^4} + F\frac{\partial^2 u^0}{\partial x^2} + \gamma \frac{\partial u^0}{\partial t} + m \frac{\partial^2 u^0}{\partial t^2} \right) \delta u \, \mathrm{d}x.$$

Therefore,

$$\delta\Phi = 0$$

when

$$\frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial u}{\partial t} + m \frac{\partial^2 u}{\partial t^2} = 0$$

with

$$u(0, t) = \frac{\partial u(0, t)}{\partial x} = \frac{\partial^2 u(1, t)}{\partial x^2} = \frac{\partial^3 u(1, t)}{\partial x^3} = 0$$

The above is obtained with an *a posteriori* subsidiary condition

$$u = u^0.$$

Further, if Φ is to be employed to form a basis of approximate solution, we may consider

$$u = \sum_n a_n(t)\varphi_n(x)$$

$$u^0 = \sum_n a_n^0(t)\varphi_n(x)$$

where $\varphi_n(x)$ are trial functions satisfying $\varphi(0) = \varphi'(0) = \varphi''(1) = \varphi'''(1) = 0$. This set of trial functions when substituted into the local potential Φ and minimized with respect to a_n yields with the subsidiary condition

$$\{a_n(t)\} = \{a_n^0(t)\}$$

a system of linear, ordinary, differential equations for $a_n(t)$. This system of equations is the same as that when the Galerkin method is used directly to obtain an approximate solution of the equation of motion of Beck's problem.

where

$$\begin{aligned} z_3 &= c_1 \cosh \xi(t-\tau)+x+a(\tau) \\ &= (t_3-\tau)(t_3^1-\tau)\gamma/2 \end{aligned} \tag{74}$$

$$\begin{aligned} z_4 &= c_1 \cosh \xi(t-\tau)+x-a(\tau) \\ &= (t_4-\tau)(\tau-t_4^1)\gamma/2 \end{aligned} \tag{75}$$

$$\begin{matrix} t_3 \\ t_3^1 \end{matrix} = [\pm\sqrt{[c_1^2 \cosh^2 \xi - 2\gamma(c_1 t \cosh \xi + x) + c_1 \cosh \xi]/\gamma}] \tag{76}$$

$$\begin{matrix} t_4 \\ t_4^1 \end{matrix} = \{\pm\sqrt{[c_1^2 \cosh^2 \xi + 2\gamma(c_1 t \cosh \xi + x) - c_1 \cosh \xi]/\gamma}\}. \tag{77}$$

The first term in (36) may now be written as

$$\begin{aligned} I_{12} &= \int_0^t G_3 H a \dot{a}\, \mathrm{d}\tau \\ &= \gamma \int_0^{t_4} (t_4-\tau)^{-\frac{1}{2}} T_2(\tau, t_3, t_3^1, t_4^1)\, \mathrm{d}\tau \end{aligned} \tag{78}$$

where

$$T_2 = T_1(\tau, t_3, t_3^1, t_4^1). \tag{79}$$

Following the procedures similar to those for I_{11}, equation (78) leads to

$$\frac{\partial I_{12}}{\partial t} = 2\gamma \int_0^{t_4} (t_4-\tau)^{\frac{1}{2}} \left[\frac{\partial t_2}{\partial t}\frac{\partial^2 T_2}{\partial \tau^2} + \frac{\partial^2 T_2}{\partial t\, \partial \tau}\right] \mathrm{d}\tau. \tag{80}$$

The integrands in equations (72) and (80) are smooth, regular functions of τ with vanishing values at both limits of integration. Physically speaking, the stress waves generated by fracturing immediately attain the speeds c_1 and c_2 while the crack propagates with a relatively small beginning speed γt. This leads to the situation that the crack size soon becomes small compared to the size of wave fronts. Therefore, it is plausible to assume $x/c_1 t \cosh \xi \ll 1$ for $x \leq a(t)$. Under this assumption, the values in equations (68) and (69) become

$$\begin{aligned} t_1 &= 2c_1 \cosh \xi/\gamma - t,\ t_1^1 = t \\ t_2 &= t \quad \text{and} \quad t_2^1 = -(2c_1 \cosh \xi/\gamma + t). \end{aligned} \tag{81}$$

In terms of the above quantities, the value of the integrand at the middle of the integration interval can be calculated. If a three-point parabolic method of integration is used, the integral (72) is evaluated as shown in Appendix D. If the similar procedures are used, the value of (80) is found as $\partial I_{12}/\partial t = \partial I_{11}/\partial t$. In terms of these quantities, the first approximation of Q_1 for an accelerating speed is found as

$$Q_{1a}^1 = \frac{p_0}{K} c_1 D_{1a} \tag{82}$$

where the constant D_{1a} is defined in Appendix D.

The integrals of wave functions for the first approximation Q_2 as in equation (43) can similarly be written as

$$I_{21} = \int_0^{\mathbf{t}_2} \mathbf{G}_2 a\dot{a}\, d\tau = \gamma \int_0^{\mathbf{t}_2} (\mathbf{t}_2-\tau)^{-\frac{1}{2}} \mathbf{T}_1(\tau, \mathbf{t}_1, \mathbf{t}_1^1, \mathbf{t}_2^1)\, d\tau \tag{83}$$

and

$$I_{22} = \int_0^{\mathbf{t}_4} \mathbf{G}_3 a\dot{a}\, d\tau = \gamma \int_0^{\mathbf{t}_4} (\mathbf{t}_4-\tau)^{-\frac{1}{2}} \mathbf{T}_2(\tau, \mathbf{t}_3, \mathbf{t}_3^1, \mathbf{t}_4^1)\, d\tau. \tag{84}$$

Integrations by parts of equations (83) and (84) followed by differentiations give

$$\frac{\partial I_{21}}{\partial x} = 2\gamma \int_0^{\mathbf{t}_2} (\mathbf{t}_2-\tau)^{\frac{1}{2}} \left[\frac{\partial \mathbf{t}_2}{\partial x}\frac{\partial^2 \mathbf{T}_1}{\partial \tau^2} + \frac{\partial^2 \mathbf{T}_1}{\partial x\, \partial \tau}\right] d\tau \tag{85}$$

and

$$\frac{\partial I_{22}}{\partial x} = 2\gamma \int_0^{\mathbf{t}_4} (\mathbf{t}_4-\tau)^{\frac{1}{2}} \left[\frac{\partial \mathbf{t}_4}{\partial x}\frac{\partial^2 \mathbf{T}_2}{\partial \tau^2} + \frac{\partial^2 \mathbf{T}_2}{\partial x\, \partial \tau}\right] d\tau. \tag{86}$$

The expressions for the differentiations of $\mathbf{T}_1$ and $\mathbf{T}_2$ are given in Appendix D. Under the assumption $x/c_2 t\eta \ll 1$, $\partial I_{21}/\partial x$ is similarly calculated as shown in Appendix D and it is found that $\partial I_{22}/\partial x = -\partial I_{21}/\partial x$. In terms of these values, the first approximation of Q_2 as in equation (44) becomes

$$Q_{2a}^1 = \frac{p_0}{K} D_{2a} \tag{87}$$

where the constant D_{2a} is defined in Appendix D. Both D_{1a} in equation (82) and D_{2a} in equation (87) are independent of x and t. It can now be seen that the successive approximations of w in equation (29) lead to an infinite series similar to that in equation (47). Therefore, the crack shape for an accelerating crack can be written as in equation (47) as

$$w_a(x, t) = \frac{p_0}{K_{Da}} \sqrt{(a^2 - x^2)} \tag{88}$$

where

$$K_{Da} = K(1+\varepsilon_a) \tag{89}$$

and

$$\varepsilon_a = (1-\nu)(k^2 D_{1a} + D_{2a}). \tag{90}$$

To find stress intensity factors and stress distribution around the accelerating crack tips $x > a(t)$, the integral Q_1 similar to that in equation (52) is first considered. In view of the regions of integration shown in Fig. 1 for an accelerating crack, the integral Q_1 for $x > a$ can be written from equations (52), (65), (68), (69) and (73) as

$$Q_{1a} = \frac{p_0}{\pi K_{Da}} \int_0^{\infty} \left[\frac{\partial I_{12}^1}{\partial t} + \frac{\partial I_{11}^1}{\partial t} - \frac{\partial I_{13}^1}{\partial t}\right] d\xi \tag{91}$$

where

$$I_{12}^{1} = \gamma \int_{0}^{t} (t_4-\tau)^{-\frac{1}{2}} T_2 \, \mathrm{d}\tau \tag{92}$$

$$I_{11}^{1} = \gamma \int_{0}^{t_2} (t_2-\tau)^{-\frac{1}{2}} T_1 \, \mathrm{d}\tau \tag{93}$$

$$I_{13}^{1} = \gamma \int_{t_1^1}^{t} (\tau-t_1^1)^{-\frac{1}{2}} T_3 \, \mathrm{d}\tau \tag{94}$$

and

$$T_3 = \tau^3[(t_1-\tau)(\tau-t_2)(\tau-t_2^1)]^{-1}. \tag{95}$$

All the three terms in equation (91) implicitly involve singularities. To obtain stress intensity factors, these singularities should be explicitly brought out. If each integral in equations (92)–(94) is first integrated by parts, then is differentiated with respect to time, and finally is integrated by parts again, the results obtained show explicit singularities at $x = a$ in the following two equations:

$$\frac{\partial I_{12}^{1}}{\partial t} = S_{12} - 2\gamma(t_4-t)^{\frac{1}{2}} \frac{\partial t_4}{\partial t} \frac{\partial T_2}{\partial t} + 2\gamma \int_{0}^{t} (t_4-\tau)^{\frac{1}{2}} \left[\frac{\partial t_4}{\partial t} \frac{\partial^2 T_2}{\partial \tau^2} + \frac{\partial T_2}{\partial \tau \, \partial t} \right] \mathrm{d}\tau \tag{96}$$

$$\frac{\partial I_{13}^{1}}{\partial t} = S_{13} + 2\gamma(t-t_1^1)^{\frac{1}{2}} \frac{\partial t_1^1}{\partial t} \frac{\partial T_3}{\partial t} - 2\gamma \int_{t_1^1}^{t} (t-t_1^1)^{\frac{1}{2}} \left[\frac{\partial t_1^1}{\partial t} \frac{\partial^2 T_3}{\partial \tau^2} + \frac{\partial^2 T_3}{\partial \tau \, \partial t} \right] \mathrm{d}\tau \tag{97}$$

and

$$\frac{\partial I_{11}^{1}}{\partial t} = 2\gamma \int_{0}^{t_2} (t_2-\tau)^{\frac{1}{2}} \left[\frac{\partial t_2}{\partial t} \frac{\partial^2 T_1}{\partial \tau^2} + \frac{\partial^2 T_1}{\partial t \partial \tau} \right] \mathrm{d}\tau, \tag{98}$$

where the singularities at $x = a$ in equations (96) and (97) are

$$\begin{aligned} S_{12} &= \left(1 - \frac{\partial t_4}{\partial t}\right) \gamma T_2(t, t_3, t_3^1, t_4^1)(t_4-t)^{-\frac{1}{2}} \\ &= \left(1 - \frac{\partial t_4}{\partial t}\right) \frac{Va}{\sqrt{(x^2-a^2)}} \end{aligned} \tag{99}$$

and

$$\begin{aligned} S_{13} &= \left(1 - \frac{\partial t_1^1}{\partial t}\right) \gamma T_3(t, t_1, t_2, t_2^1)(t-t_1^1)^{-\frac{1}{2}} \\ &= \left(1 - \frac{\partial t_1^1}{\partial t}\right) \frac{Va}{\sqrt{(x^2-a^2)}}. \end{aligned} \tag{100}$$

The integral Q_{2a} for $x > a$ may be obtained by applying the similar procedures for Q_{1a} to equation (58). The results may be written as

$$Q_{2a} = \frac{p_0}{2K_{Da}}\left[\frac{\partial I_{22}^1}{\partial x} - \frac{\partial I_{21}^1}{\partial x} + \frac{\partial I_{23}^1}{\partial x}\right] \tag{101}$$

$$\begin{aligned} \frac{\partial I_{22}^1}{\partial x} &= \gamma\frac{\partial}{\partial x}\int_0^t (\mathbf{t}_4-\tau)^{-\frac{1}{2}}\mathbf{T}_2\, d\tau \\ &= S_{22} - 2\gamma(\mathbf{t}_4-t)^{\frac{1}{2}}\left[\frac{\partial \mathbf{T}_2}{\partial x} + \frac{\partial \mathbf{t}_4}{\partial x}\frac{\partial \mathbf{T}_2}{\partial t}\right] + 2\gamma\int_0^t (\mathbf{t}_4-\tau)^{\frac{1}{2}}\left[\frac{\partial t_4}{\partial x}\frac{\partial^2 \mathbf{T}_2}{\partial \tau^2} + \frac{\partial^2 \mathbf{T}_2}{\partial \tau\, \partial x}\right] d\tau \end{aligned} \tag{102}$$

and

$$\frac{\partial I_{23}^1}{\partial x} = S_{23} + 2\gamma(t-t_1^1)^{\frac{1}{2}}\left[\frac{\partial \mathbf{T}_3}{\partial x} + \frac{\partial t_1^1}{\partial x}\frac{\partial \mathbf{T}_3}{\partial t}\right] - 2\gamma\int_{t_1^1}^t (\tau-t_1^1)^{\frac{1}{2}}\left[\frac{\partial t_1^1}{\partial x}\frac{\partial^2 \mathbf{T}_3}{\partial \tau^2} + \frac{\partial^2 \mathbf{T}_3}{\partial \tau\, \partial x}\right] d\tau \tag{103}$$

where the singularities at $x = a$ are

$$S_{22} = -\gamma\frac{\partial \mathbf{t}_4}{\partial x}(\mathbf{t}_4-t)^{-\frac{1}{2}}\mathbf{T}_2 = -\frac{(\partial \mathbf{t}_4/\partial x)Va}{\sqrt{(x^2-a^2)}} \tag{104}$$

and

$$S_{23} = -\gamma\frac{\partial \mathbf{t}_1^1}{\partial x}(t-\mathbf{t}_1^1)^{-\frac{1}{2}}\mathbf{T}_3 = -\frac{(\partial \mathbf{t}_1^1/\partial x)Va}{\sqrt{(x^2-a^2)}}. \tag{105}$$

If equations (51), (91) and (101) are substituted in equations (22) and (31), the resulting normal stresses σ_{yy} and σ_{xx} can be written as sums of singular terms and regular functions which can be calculated numerically for various values of x and t. The stress intensity factors are obviously concerned with these singular terms. In the cleavage plane, the dynamic stress intensity correction factor obtained from equations (22), (51), (91) (101) and (62) can be written as

$$K_{Ia} = \frac{1+\varepsilon}{1+\varepsilon_a}K_{Iv}. \tag{106}$$

The dynamic stress intensity correction factor in the direction normal to the crack plane is found from equations (31), (32), (51), (91), (101) and (64) as

$$K_{Ia}^1 = \frac{1+\varepsilon}{1+\varepsilon_a}K_{Iv}^1. \tag{107}$$

The dynamic stress intensity factors in the cleavage and noncleavage planes for an accelerating crack are respectively equal to the above two constants multiplied by the associated static stress intensity factors such as $K_{Ia}p_0\pi a$ and $K_{Ia}^1 p_0\pi a$.

5. DISCUSSIONS AND CONCLUSIONS

Exact expressions are obtained for the stresses and the crack shape in the plane of a crack which propagates at a constant speed V. The value of the dynamic correction factor $(1+\varepsilon)^{-1}$ for the crack shape in equation (47) is calculated by using an electronic computer for $\nu = 0{\cdot}25$ and various values of V/c_2, as shown in Fig. 2. In the same figures, the curves for

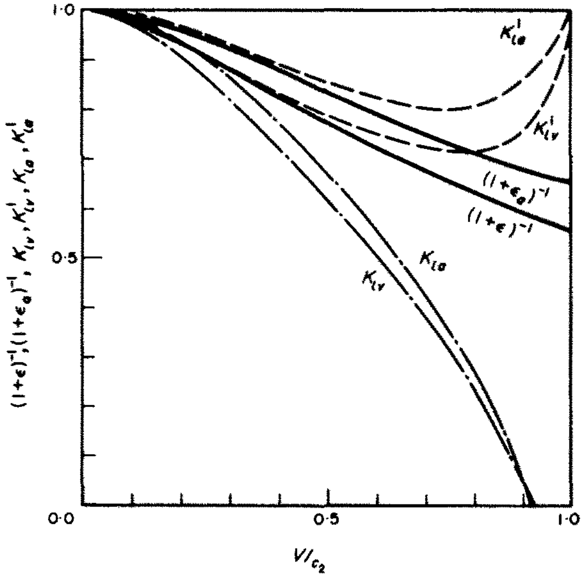

FIG. 2. Nondimensional dynamic correction factors for constant speed and accelerating cracks.

the dynamic stress intensity factors K_{Iv} and K_{Iv}^1 are also shown as a function of V/c_2. The calculations of the above curves involve the operators L_1 and $\bar{L}_1$ as defined in Appendix A. These operators have singularities at $\eta = k$ which are removed during computations by introducing the transformation $\eta^2 = (k^2-1)\cos h^2\theta + 1$. In terms of the new variable θ, the integrals of the operators become regular and can be evaluated by a regular four-point integration method in the computer. The terminal crack speed for brittle materials was found to be $V = 0{\cdot}38\,(E/\rho)^{\frac{1}{2}} = 0{\cdot}6\,c_2$ for $\nu = 0{\cdot}25$ [13]. For this crack speed, $w/w_0 = (1+\varepsilon)^{-1} = 0{\cdot}724$ as seen in Fig. 2. In other words, the deformation at the surface of a dynamic crack running at its terminal speed is 27·6 per cent smaller than the corresponding static crack deformation.

The normal stresses σ_{xx} and σ_{yy} in the plane of a static plane crack were found to be identical and a state of hydrostatic tension was shown to exist in the crack plane [14]. This is a limiting case of the current result which is obtained by letting $V = 0$ in equations (61) and (63). If the speed V differs from zero, the values of K_{Iv} and K_{Iv}^1 are different as seen in Fig. 2. Therefore, the state of hydrostatic tension ceases to exist, once the crack propagates. The value of K_{Iv} shown in Fig. 2 decreases with increasing V/c_2 and becomes zero at $V = 0{\cdot}92\,c_2$ which is the Rayleigh wave speed for $\nu = 0{\cdot}25$. This value may be suggested as the maximum possible value of the crack speed. The value of K_{Iv}^1 is consistently higher than K_{Iv}. It first decreases for small V/c_2 and increases back to a relative high value for large V/c_2.

The crack shape and the stress distribution in the region small compared to the wave fronts are obtained for a crack propagating at a constant acceleration. The values of dynamic corrections factors such as $(1+\varepsilon_a)^{-1}$, K_{Ia} are obtained as shown in Fig. 2. These values are consistently higher than their corresponding values for a constant-speed crack. In other words, the deviations of stresses and deformations from the associated state state are smaller for an accelerating crack than for a constant-speed crack propagating at the same speed V.

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APPENDIX A

Operators

$$L_1 \cos(sc_2\eta t) = \frac{8}{\pi}\int_1^k (\eta^2-1)^{\frac{1}{2}}\eta^{-3}\cos(sc_2\eta t)\,\mathrm{d}\eta$$

$$+\frac{8}{\pi}\int_k^\infty [1-\eta^2+(\eta^2k^{-2}-1)^{\frac{1}{2}}(\eta^2-1)^{\frac{1}{2}}]\eta^{-3}(\eta^2k^{-2}-1)^{-\frac{1}{2}}\cos(sc_2\eta t)\,\mathrm{d}\eta$$

$$\bar{L}_1 \cos(sc_2\eta t) = \frac{8}{\pi}\int_k^\infty [\eta^2k^{-2}-1-(\eta^2k^{-2}-1)^{\frac{1}{2}}(\eta^2-1)^{\frac{1}{2}}]\eta^{-3}(\eta^2k^{-2}-1)^{-\frac{1}{2}}\cos(sc_2\eta t)\,\mathrm{d}\eta$$

$$-\frac{8}{\pi}\int_1^k (\eta^2-1)^{\frac{1}{2}}\eta^{-3}\cos(sc_2\eta t)\,\mathrm{d}\eta$$

$$L_2 ⨍(\cosh\xi) = \frac{2}{\pi}\int_0^\infty \frac{v_2^2\cosh^2\xi}{(\cosh\xi^2-v_1^2)^{\frac{3}{2}}} ⨍(\cosh\xi)\,\mathrm{d}\xi$$

$$L_3 ⨍(\eta) = L_1\frac{v_2^2\eta}{(\eta^2-v_2^2)^{\frac{3}{2}}} ⨍(\eta)$$

APPENDIX B

Formulae for integrations

(a) $$F_1 = \int_{\tau_1}^{\tau_2} \frac{\tau\,\mathrm{d}\tau}{\sqrt{[(a-b\tau)(a-\mathrm{d}\tau)]}}, \qquad d > b$$

Let $\xi = \sqrt{(a-\mathrm{d}\tau)}$, $\xi = \xi_1$ at $\tau = \tau_1$, $\xi = \xi_2$ at $\tau = \tau_2$.

$$F_1 = \frac{2}{d\sqrt{(bd)}}\left\{a\frac{d+b}{2b}\ln(\xi+\sqrt{[\xi^2+a(d/b-1)]}-\frac{\xi}{2}\sqrt{[\xi^2+a(d/b-1)]}\right\}\Bigg|_{\xi_2}^{\xi_1}$$

(b) $$F_2 = \int_{a/d}^{t} \frac{\tau\,\mathrm{d}\tau}{\sqrt{[(b\tau-a)(\mathrm{d}\tau-a)]}}, \qquad b > d$$

Let $\xi = \sqrt{(\mathrm{d}\tau-a)}$, $\xi = \xi_0$ at $\tau = t$

$$F_2 = \frac{2}{d\sqrt{(bd)}}\left\{\frac{\xi}{2}\sqrt{[\xi^2+a(1-d/b)]}+\frac{a(b+d)}{2b}\ln(\xi+\sqrt{[\xi^2+a(1-d/b)]}\right\}\Bigg|_0^{\xi_0}.$$

APPENDIX C

Dynamic constants

$$v_1 = V/c_1$$

$$v_2 = V/c_2$$

$$D_1 = \frac{2}{\pi}\int_0^\infty \frac{\cosh \xi v_1^2}{(\cosh^2 \xi - v_1^2)^{\frac{3}{2}}}\left\{2\cosh \xi \ln\left[\sqrt{\left(\frac{\cosh \xi - v_1}{2v_1}\right)} + \sqrt{\left(\frac{\cosh \xi + v_1}{2v_1}\right)}\right] - (\cosh^2 \xi \quad v_1^2)^{\frac{1}{2}}\right\} \mathrm{d}\xi$$

$$D_2 = L_1 \frac{v_2^2}{(\eta^2 - v_2^2)^{\frac{3}{2}}}\left\{2\eta \ln\left[\sqrt{\left(\frac{\eta - v_2}{2v_2}\right)} + \sqrt{\left(\frac{\eta + v_2}{2v_2}\right)}\right] - (\eta^2 - v_2^2)^{\frac{1}{2}}\right\}$$

$$\varepsilon = (1-\nu)(k^2 D_1 + D_2)$$

$$D_3 = \frac{2}{\pi}\int_0^\infty \frac{v_2^2 \cosh y}{\cosh^2 y - v_1^2}\,\mathrm{d}y = v_2^2(1 - v_1^2)^{-\frac{1}{2}}$$

$$D_4 = L_1 \frac{v_2^2}{\eta^2 - v_2^2}.$$

APPENDIX D

Functions for accelerating crack

$$Z_{11}(\tau, t_1, t_1^1, t_2^1) = [(\tau - t_2^1)(t_1 - \tau)(t_1^1 - \tau)]^{-\frac{1}{2}}$$

$$Y_{11}(\tau, t_1, t_1^1, t_2^1) = (t_1 - \tau)^{-1} + (t_1^1 - \tau)^{-1} - (\tau - t_2^1)^{-1}$$

$$\frac{\partial^2 T_1}{\partial \tau^2} = \tau(6 + \tau Y_{11})(1 - \tau Y_{11}/4)Z_{11} + \tau^2[t_2^1(\tau - t_2^1)^{-2} + t_1(t_1 - \tau)^{-2} + t_1^1(t_1^1 - \tau)^{-2}]Z_{11}/2$$

$$\frac{\partial^2 T_1}{\partial t\,\partial \tau} = -\frac{\tau^2}{4}(6 + \tau Y_{11})Z_{11}\left[\frac{\partial t_1}{\partial t}(t_1 - \tau)^{-1} + \frac{\partial t_1^1}{\partial t}(t_1^1 - \tau)^{-1} - \frac{\partial t_2^1}{\partial t}(\tau - \tau_2^1)^{-1}\right] - \tau Z_{11}\left[\frac{\partial t_1}{\partial t}(t_1 - \tau)^{-2} + \frac{\partial t_1^1}{\partial t}(t_1^1 - \tau)^{-2} + \frac{\partial t_2^1}{\partial t}(\tau - t_2^1)^{-2}\right]$$

$$\frac{\partial I_{11}}{\partial t} = c_1 v_1^2(69 - 97v_1/\cosh y - 53v_1^2/\cosh^2 y + 223v_1^3/2\cosh^3 y + 2623v_1^4/256\cosh^4 y - 10657v_1^5/256\cosh^5 y)(1 - 9v_1^2/16\cosh^2 y)^{-\frac{3}{2}}/48\cosh y$$

$$\frac{\partial^2 \mathbf{T}_1}{\partial x\,\partial \tau} = -\frac{\tau^2}{4}(6 + \tau Y_{11})Z_{11}\left[\frac{\partial \mathbf{t}_1}{\partial x}(\mathbf{t}_1 - \tau)^{-1} + \frac{\partial t_1^1}{\partial x}(\mathbf{t}_1^1 - \tau)^{-1} - \frac{\partial \mathbf{t}_2^1}{\partial x}(\tau - \mathbf{t}_2^1)^{-1}\right] - \tau \mathbf{Z}_{11}\left[\frac{\partial \mathbf{t}_1}{\partial x}(\mathbf{t}_1 - \tau)^{-2} + \frac{\partial \mathbf{t}_1^1}{\partial x}(\mathbf{t}_1^1 - \tau)^{-2} + \frac{\partial \mathbf{t}_2^1}{\partial x}(\tau - \mathbf{t}_2^1)^{-2}\right]$$

$$\frac{\partial I_{21}}{\partial x} = -v_2^2(69-97v_2/\eta-53v_2^2/\eta^2+223v_2^3/2\eta^3+2623v_2^4/256\eta^4$$
$$-10657v_2^5/256\eta^5)(1-9v_2^2/16\eta^2)^{-\frac{5}{2}}/48\eta^2$$

$$D_{1a} = \frac{2}{\pi c_1}\int_0^\infty \frac{\partial I_{11}}{\partial t}\,\mathrm{d}\xi$$

$$D_{2a} = -L_1\frac{\partial I_{21}}{\partial x}.$$

REFERENCES

[1] B. COTTERELL, Velocity effects in fracture propagation. *Applied Materials Research* **4**, 227 (1965).
[2] E. N. DULANLY and W. F. BRACE, Velocity behavior of a growing crack. *J. appl. Phys.* **31**, 2233 (1960).
[3] G. C. SIH and H. LIEBOWITZ, Mathematical Theories of Brittle Fracture. *Fracture* (Edited by H. LIEBOWITZ), pp. 67–190. Academic Press (1964).
[4] Y. M. TSAI, Exact Stress Distribution, Crack Shape and Energy for a Running Penny-Shaped Crack in an Infinite Elastic Solid. Presented to the Fifth National Symposium on Fracture Mechanics held at University of Illinois, 1971 (To appear in *Int. J. Fract. Mech.*).
[5] K. B. BROBERG, The propagation of a brittle crack. *Arkiv for Fysik* **18**, 159 (1960).
[6] I. N. SNEDDON, *Fourier Transforms*. p. 429. McGraw-Hill (1951).
[7] Y. M. TSAI, Dynamic contact stresses produced by the impact of an axisymmetrical projectile on an elastic half-space. *Int. J. Solids Struct.* **7**, 543 (1971).
[8] Y. M. TSAI, Stress waves produced by impact on the surface of a plastic medium. *J. Franklin Inst.* **285**, 204 (1968).
[9] C. L. PEKERIS, The seismic surface pulse. *Proc. Nat. Acad. Sci.* **41**, 469 (1955).
[10] H. LAMB, On the propagation of tremors over the surface of an elastic solid. *Phil. Trans.* **A-203**, 1–42 (1904).
[11] Y. M. TSAI, Stress distribution, crack shape and energy for a penny-shaped crack in a plate of finite thickness. *Eng. Fract. Mech.* **4**, 155 (1972).
[12] Y. M. TSAI, Stress distribution in elastic and viscoelastic plates subjected to symmetrical rigid indentations. *Q. appl. Math.* **27**, 371 (1961).
[13] D. K. ROBERTS and A. A. WELLS, The velocity of brittle fracture. *Engineering* **178**, 820 (1954).
[14] M. L. WILLIAMS, On the stress distribution at the base of a stationary crack. *J. appl. Mech.* 109 (1957).
[15] G. C. SIH, Dynamic Aspects of Crack Propagation. *Inelastic Behavior of Solids* (Edited by KANNINEN *et al.*), p. 607. McGraw-Hill (1969).
[16] J. W. CRAGGS, Fracture criteria for use in continuum mechanics. *Fracture of Solids* (Edited by DRUCKER and GILMAN) p. 51. Interscience (1963).
[17] J. D. ESHELBY, The elastic field of a crack extending nonuniformly under general anti-plane loading. *J. Mech. Phys. Solids* **17**, 177 (1969).
[18] G. I. BARENBLATT and G. P. CHERAPANOV, On brittle cracks under longitudinal shear. *Prikl. Mat. i Mek.* **24**, 1654 (1961).
[19] B. V. KOSTROV, The axisymmetric problem of propagation of a tension crack. *Prikl. Mat. i Mek.* **28**, 793 (1964).

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Абстракт—Получаются интегральные решения для симметрической трещины, которая распространяется с изменяющейся скоростью в упругом твердом теле, под влиянием действия произвольного симметрического давления на трещину, в виде суммы присоединенных статических решений и интегралов волны напряжения. Исследуются подробно два случая. Для случая симметрической трещины, движущейся с постоянной скоростью, под влиянием постоянного давлення, даются точные динамические решения в замкнутом виде дЛя формы трещины и распределения напряжений с сингулярностями в плоскости трещины, что дает возможность сравнения с присоединенными статическими решениями. Определяется разница между динамическими и статическими решениями, для величин такнх как форма трещины и факторы интенсивности напряжений, путем факторов динамической поправки. Эти факторы представляют собой безразмерные функции коэффициента Пуассона и соотнешения между скоростями трещины и волны сдвига. Получаются значения этих динамических факторов для широкого предела скорости трещины. На основе полученных результатов можно легко определить разницу.

Приводится исследование для распространения трещины с постоянным ускорением. Затем, получаются подобные величины к таким-же, определенным для указанной выше трещины с постоянной скоростью. Отклонения факторов формы трещины и интенсивности напряажений от присоединенного статического состояния оказывается, относительно, меньшими лдя трещины с ускорением по сравнению с трещиной с постоянной скоростью, для распространения с такой же самой скоростью.